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A Measure of the Randomness of Signals and Suggested Applications

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FOR THE COMMANDER

Thomas J. Alpert

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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A MEASURE OF THE RANDOMNESS OF SIGNALS AND SUGGESTED APPLICATIONS*

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Division 6

TECHNICAL REPORT 750

7 MARCH 1986

*This report is an edited version of a Laboratory seminar delivered by the author on 14 February 1986. The format displays the text associated with the vugraphs used during the seminar.

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ABSTRACT

The solution to an optimization or estimation problem depends on some criterion of optimality. Typically, all prospective solutions have an associated measure of goodness, and the optimum solution is that with maximum (or minimum) measure. The choice of measure incorporates whatever is known about the physics of the problem as well as other assumptions.

In this presentation, we suggest two closely related measures of the randomness of signals. Each of these measures is related to the uncertainty principle of quantum mechanics. We further suggest that minimizing either of these measures is an appropriate criterion for the solution of a variety of problems, including spectral estimation, signal extrapolation and/or interpolation, non-parametric signal detection, and phase retrieval. The criterion may be appropriate when the unknown signal is postulated to be "simple" rather than "complex" and we want to minimize the number of arbitrary assumptions made about the signal and noise characteristics.

The outline of this paper is as follows. First I will try to motivate why it is desirable to quantify the randomness of signals. I will then take what appears to be a digression by discussing ambiguity functions and entropy. This discussion will give rise to two versions of the well-known uncertainty principle of quantum mechanics, which is formally identical to an uncertainty principle of signals. The two forms of uncertainty principle suggest two measures of the randomness of signals. I will then discuss in broad outline how these measures can be used in solving problems of general interest. The paper closes with some final remarks.

A MEASURE OF THE RANDOMNESS OF SIGNALS AND SUGGESTED APPLICATIONS

OUTLINE

- MOTIVATION
- AMBIGUITY FUNCTIONS AND THEIR TRANSFORMS
- ENTROPY
- TWO VERSIONS OF THE UNCERTAINTY PRINCIPLE
- TWO MEASURES OF RANDOMNESS
- POTENTIAL APPLICATIONS USING MINIMUM RANDOMNESS AS OPTIMIZATION CRITERION
- FINAL REMARKS

73732-1

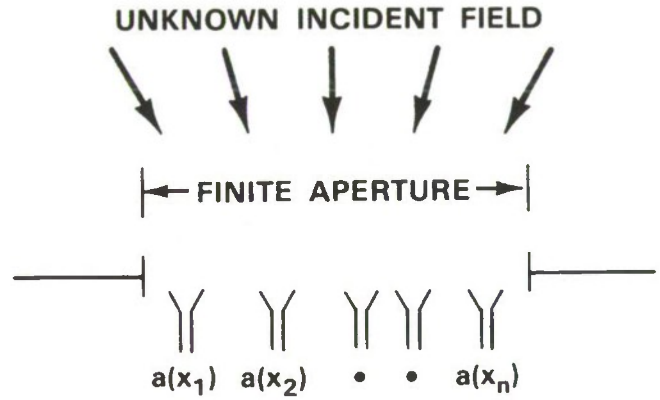
MOTIVATION

Consider the following problem that's familiar to many people at the Laboratory. Suppose there is an unknown electromagnetic field incident upon an antenna with finite aperture. The field is sampled at several points along the aperture. What we want to do is estimate the unknown field which gave rise to the observed samples.

I want to make some observations about this problem. The first thing to observe is that the problem is incompletely specified. Said a little differently, there are many aperture illumination functions, $U(\theta)$, that are compatible with the available data. So the real question is, "How do I pick a specific $U(\theta)$ that I'm going to use as my estimate; how do I distinguish a good $U(\theta)$ from a bad one?" Implicit in asking that question is the existence of some criterion to compare two different $U(\theta)$ to determine which is preferable. Said a little more precisely, what that really means is that for every $U(\theta)$ there exists a real number or measure, M. If the number is small, the corresponding $U(\theta)$ is good and if the number is big, the corresponding $U(\theta)$ is bad. Optimization entails picking the $U(\theta)$ with the smallest M.

When one picks a measure for use in comparing various $U(\theta)$, that choice implies assumptions about structure within the problem. Sometimes that structure or the assumptions are very plausible and reasonable; i.e., they fit the physics of the problem. However, sometimes they are arbitrary. They are put in because it makes the problem easy to solve. The only reason I emphasize this point is because when we select a measure we should be clear about what we're assuming and how the assumptions affect the problem solution.

TYPICAL PROBLEM



GIVEN: SAMPLES OF APERTURE ILLUMINATION $\{a(x_i)\}$

ESTIMATE: UNKNOWN INCIDENT FIELD $U(\theta)$

OBSERVATIONS

PROBLEM IS INCOMPLETELY SPECIFIED; i.e., MANY $U(\theta)$ ARE COMPATIBLE WITH OBSERVED $\{a(x_i)\}$.

CHOICE AMONG VARIOUS ADMISSIBLE $U(\theta)$ IMPLIES EXISTENCE OF SOME CRITERION TO CHOOSE AMONG THEM; i.e., CHOOSE $U(\theta)$ WITH MINIMUM MEASURE M[U]

SPECIFICATION OF MEASURE IMPLIES ADDITIONAL STRUCTURE AND/OR ASSUMPTIONS

73732-2

I want to dwell for a moment on the subject of choosing measures. What do we want a measure to do or what properties do we want it to have?

The first one is subjective: it should be pleasing. I guess you can't say too much more about that except to observe that this property is subjective.

I'd like to make a strong point that the measure should incorporate a minimum number of assumptions. It shouldn't imply any more structure in the problem than is really there. To the extent that it contains more than that, those assumptions are arbitrary and certainly that's going to shape the solution. If the assumptions are arbitrary, the solution is going to be arbitrary. So, you don't want your measure to have more assumptions than you are really entitled to.

It wants to be robust in the sense that if you change its argument a little bit, you don't want the measure to change very much.

It should be tractable. When we're given the samples $\{a(x_i)\}$ we want to turn a crank to get $U(\theta)$ and we don't want that crank to be too long. We don't want to burn up too much computer time or whatever to accomplish the solution.

Another major desirable attribute is that after we turn this crank, we want the solution to be unique. It's a little disconcerting to set up the problem and find there are ten different $U(\theta)$ that minimize the measure. How do we choose among them? We have no basis for saying that one is better than the other if they all have the same measure.

DESIRABLE ATTRIBUTES OF MEASURE USED IN OPTIMIZATION PROBLEMS

"INTUITIVELY PLEASING"

MINIMUM ARBITRARY ASSUMPTIONS

ROBUST — M[U] CONTINUOUS IN U

TRACTABLE — DERIVING $U(\theta)$ FROM $\{a(x_i)\}$ SHOULD NOT BE TOO DIFFICULT

UNIQUE SOLUTION

73732-3

The ambiguity function as a waveform u(t) is simply the cross-correlation between the time-shifted version of u(t) [$u(t + \tau)$] and the frequency-shifted version of u(t) [$u(t) \exp(-j2\pi\phi t)$]. The ambiguity function can also be expressed as an integral in the frequency domain. This is a well-known formulation generally credited to Woodward in 1955.

AMBIGUITY FUNCTION

AMBIGUITY FUNCTION IS CROSS CORRELATION BETWEEN TIME AND FREQUENCY SHIFTED VERSIONS OF THE SAME WAVEFORM

	WAVEFORM	TIME SHIFT = τ	FREQ SHIFT = ϕ
TIME DOMAIN	$u(t)$	$u(t+\tau)$	$u(t)\ e^{-j2\pi\phi t}$
FREQ DOMAIN	$U(f)$	$U(f)\ e^{j2\pi f\tau}$	$U(f+\phi)$

$$A_W(\tau,\phi) = \int u(t)\ u^*(t+\tau)\ e^{-j2\pi\phi t}dt$$

$$= \int U^*(f)\ U(f+\phi)\ e^{-j2\pi f\tau}df$$

(WOODWARD, 1955)

73732-21

The ambiguity function has a couple of interesting properties. If we normalize the waveforms to have unit energy, the ambiguity function evaluated at $\tau = 0$, $\phi = 0$, equals 1. The volume of its square magnitude integrated over its two variables is one. Therefore, in geometric terms, the ambiguity function of every normalized u(t) has constant volume over the (τ,ϕ) plane, with maximum value at the origin, and falling off away from the origin. How it falls off depends on u(t), but if you try to design u(t) to have a small magnitude of A at some (τ,ϕ) its magnitude will pop up somewhere else. This is a qualitative description of ambiguity between time (delay) and frequency (Doppler) that radar people are very familiar with.

The two-dimensional Fourier transform of the ambiguity function is simply the product of the time waveform, its spectrum, and a phase factor. Note that this transform is in general a complex number.

PROPERTIES OF $A_W(\tau,\phi)$

FOR $\int |u(t)|^2 = 1 = \int |U(f)|^2$

- $A_W(0,0) = 1$
- $\int |A_W(\tau,\phi)|^2 \, d\tau \, d\phi = 1$

GEOMETRIC INTERPRETATION: VOLUME UNDER $|A_W|^2$ CONTOUR IS A CONSTANT FOR ALL u(t)

TWO DIMENSIONAL FOURIER TRANSFORM IS

$$F_W(p,q) = u(p) \; U^*(q) \; e^{-j2\pi pq}$$

73732-22

The Woodward ambiguity function that I just described is one of many forms of the ambiguity function that can be defined. I'm going to select a slightly different version of the ambiguity function because we're going to do more interesting things with it. If the time function u(t) is shifted in time by $\pm\tau/2$ and shifted in frequency by $\pm\phi/2$, the set of relationships indicated in the table is obtained. I then proceed to define an ambiguity function as the cross correlation between the waveform shifted by $\tau/2$ and $\phi/2$, with the same waveform shifted by $-\tau/2$ and $-\phi/2$. This defines another ambiguity function in the same spirit as the first, but a different function. It, too, is representable as a time integral or a frequency integral.

ALTERNATE AMBIGUITY FUNCTION

	WAVEFORM	TIME SHIFT $\pm \frac{\tau}{2}$	FREQ SHIFT $\pm \frac{\phi}{2}$	BOTH SHIFTS
TIME DOMAIN	$u(t)$	$u\left(t \pm \frac{\tau}{2}\right)$	$u(t)\ e^{\mp j\pi\phi t}$	$u\left(t \pm \frac{\tau}{2}\right) e^{\mp j\pi\phi t}$
FREQ DOMAIN	$U(f)$	$U(f)\ e^{\pm j\pi f\tau}$	$U\left(f \pm \frac{\phi}{2}\right)$	$U\left(f \pm \frac{\phi}{2}\right) e^{\pm j\pi f\tau}$

$$A_M(\tau,\phi) = \int \left[u\left(t + \frac{\tau}{2}\right)e^{+j\pi\phi t}\right]\left[u\left(t - \frac{\tau}{2}\right)e^{-j\pi\phi t}\right]^* dt$$

$$= \int u\left(t + \frac{\tau}{2}\right) u^*\left(t - \frac{\tau}{2}\right) e^{j2\pi\phi t}\ dt$$

$$= \int U^*\left(f + \frac{\phi}{2}\right) U\left(f - \frac{\phi}{2}\right) e^{j2\pi f\tau}\ df$$

73732-19

A_M has interesting Fourier transform properties. If we proceed to take the two-dimensional Fourier transform of A_M in dummy variables p and q, we get an expression which, in fact, is a convolution.

This transform has the interesting property of being a real function. It's very easy to prove this fact by showing that $F = F^*$. F has the property that if you integrate on p you get the power spectrum, and if you integrated on q you get the instantaneous power. Integration on both variables yields unity. All of these attributes are essentially identical to what we've come to know and love as an "honest-to-God" two-dimensional probability density. The only place in which it might deviate from a real probability density is that F(p,q), although real, may be negative.

This particular density function, $F_M(p,q)$, was first discussed by the Nobel-prize winning physicist Eugene Wigner in 1932, and revisited by Moyal in 1949. I distinguish the Wigner-Moyal distribution F_M from the corresponding Woodward function F_W by subscript. Wigner and Moyal considered F_M in the context of quantum mechanics in which the function u(p) was a wave function expressed in p coordinates, and its Fourier transform U(q) was the same wave function expressed in q coordinates, where p and q are conjugate variables.

Moyal revisited the Wigner function and observed that its two-dimensional transform A_M is in fact a moment-generating function of F_M viewed as a probability density. Therefore, we can proceed to calculate all the moments of this density.

2-DIMENSIONAL FOURIER TRANSFORM OF $A_M(\tau, \phi)$

$$F_M(p,q) = \int u\left(p - \frac{r}{2}\right) u^*\left(p + \frac{r}{2}\right) e^{j2\pi qr}\, dr$$

$$\int U^*\left(q - \frac{\sigma}{2}\right) U\left(q + \frac{\sigma}{2}\right) e^{j2\pi p\sigma}\, d\sigma$$

- $F_M(p,q)$ IS REAL ($F_M = F_M^*$)
- $\int F_M(p,q)\, dp = |U(q)|^2$
- $\int F_M(p,q)\, dq = |u(p)|^2$
- $\int \int F_M(p,q)\, dpdq = 1$

F_M IS WIGNER-MOYAL DENSITY FUNCTION (Wigner, 1932; Moyal, 1949)

A_M IS MOMENT GENERATING FUNCTION FOR WMD

73878-1

Moyal and the English statistician Bartlett calculated the moments of the distribution $F_M(p,q)$ using the moment-generating function A_M. More specifically, they determined the variance and covariance of the variables p and q. These quantities were shown to satisfy the inequality shown. This inequality is a statement of the uncertainty principle of quantum mechanics (with Planck's constant h set equal to unity). This form of the uncertainty principle is stronger than the more familiar for which does not include the covariance term σ_{pq}.

The uncertainty inequality is satisfied with the equal sign if u(t) is a Gaussian chirp. A Gaussian chirp, as you know, is a waveform with a Gaussian envelope and a linearly changing, instantaneous frequency. I've written u(t) centered at t = 0. If it were centered somewhere else or if the chirp were centered at some center frequency, the uncertainty inequality would still be satisfied with the equal sign. It is this observation that gives the strongest justification for the commonly held view of radar people that a Gaussian chirp is the optimum signal to use for the simultaneous localization of delay and doppler of a point target. You can get better localization of delay, but only at the expense of worse localization of doppler. This point is known intuitively by radar people, but it's interesting to see its formal and direct relationship to a very tight and specific form of the uncertainty principle of quantum mechanics.

UNCERTAINTY PRINCIPLE

$$\sigma_p^2 \ \sigma_q^2 - \sigma_{pq}^2 \geqslant \left(\frac{1}{4\pi}\right)^2 \qquad (h = 1)$$

(Bartlett and Moyal, 1949)

- STRONGER FORM THAN USUALLY SEEN
- EQUALITY FOR u(t) A GAUSSIAN CHIRP

$$u(t) = \underbrace{(2\pi\sigma^2)^{-1/4} \ \exp\left(- \frac{t^2}{4\sigma^2}\right)}_{\text{GAUSSIAN ENVELOPE}} \ \underbrace{\exp\,(j\beta t^2)}_{\text{LINEAR CHIRP}}$$

73878-2

I'm going to change themes just briefly. In this viewgraph I've summarized some material that's familiar to information theorists. For any multi-dimensional probability density, we can define a function called entropy which is simply the integral of $-R \log R$. If we particularize this to a two-dimensional probability density, $F(p,q)$, we can talk about the entropy of that two-dimensional density.

We can also talk about a second quantity; the mutual information $I(P,Q)$ between the two variables, p and q. $I(P,Q)$ is never negative. The mutual information can be expressed as the entropy of the P density, plus the entropy of the Q density, less the entropy of the joint density. The only purpose for doing all of this is to get the final inequality which states that the entropy of P plus the entropy of Q is lower bounded by the joint entropy $H(F)$.

ENTROPY AND MUTUAL INFORMATION

FOR $R(\bar{x}) \geqslant 0, \int R(\bar{x})\, d\bar{x} = 1$

ENTROPY: $H(\bar{R}) = -\int R(\bar{x}) \log R(\bar{x})\, d\bar{x}$

FOR 2-DIMENSIONAL PROBABILITY DENSITY $F(p,q)$

MUTUAL INFO: $I(P;Q) = \int \int F(p,q) \log \frac{F(p,q)}{P(p)\, Q(q)}\, dp\, dq$

WHERE $P(p) = \int F(p,q)\, dq$

$Q(q) = \int F(p,q)\, dp$

$I(P;Q) = H(P) + H(Q) - H(F) \geqslant 0$

$H(P) + H(Q) \geqslant H(F)$

73732-16

We make one final digression before we tie all of this together. Consider the following problem. Which two-dimensional probability function F(p,q) with specified values of σ_p^2, σ_q^2, and σ_{pq}, has minimum entropy?

The seemingly obscure problem was addressed by Leipnik in 1960. He solved it and, not surprisingly, showed that the bi-variate Gaussian is the answer. He found that the minimum entropy is a fairly complicated expression which is a function only of the determinant of the covariance matrix. If F happens to be a Wigner-Moyal distribution, we know that the determinant of the covariance matrix is no smaller than $\pi^2/4$. Utilizing this result in Leipnik's minimization problem implies that the minimum entropy obtainable from any Wigner-Moyal distribution is log e/2. The entropy of F derived from a Gaussian chirp equals the minimum value.

A MINIMIZATION PROBLEM

WHICH TWO DIMENSIONAL PROBABILITY DENSITY $F(p,q)$ WITH SPECIFIED COVARIANCE MATRIX Σ HAS MINIMUM ENTROPY?

SOLUTION: BIVARIATE GAUSSIAN

$$H_{MIN}(F) = 1 + \frac{1}{2} \log(4\pi^2 \det \Sigma)$$

FOR F A WIGNER-MOYAL DISTRIBUTION

$$\text{DET } \Sigma = \sigma_p^2 \sigma_q^2 - \sigma_{pq}^2 \geqslant \left(\frac{1}{4\pi}\right)^2$$

$$H_{MIN}(F) \geqslant \log \frac{e}{2}$$

(Leipnik, 1960)

73732-15

The stage is set to state the familiar uncertainty principle in a second way. Combining our uncertainty inequality with Leipnik's result, we obtain the uncertainty principle expressed in entropy form (Eq. 1). For comparison, we express in Eq. (2) the previously stated more familiar version of the uncertainty principle in a manner parallel to the entropy form.

Having discussed lower bounds to the left-hand sides of Eqs. 1 and 2, let us consider whether there are upper bounds. The answer is no. One can construct distributions which have arbitrarily large values of $\log \sigma_p + \log \sigma_q$ or $H(P) + H(Q)$.

ALTERNATE VERSIONS OF UNCERTAINTY PRINCIPLE

(1) $H(P) + H(Q) \geqslant H(F) \geqslant \log \left(\frac{e}{2}\right)$

(2) $\log(\sigma_p) + \log(\sigma_q) \geqslant \frac{1}{2} \log(\det \Sigma) \geqslant \log \left(\frac{1}{4\pi}\right)$

<u>N.B.</u> THERE IS NO UPPER BOUND TO THE LHS OF EQUATIONS (1) OR (2)

73732-14

The stage is now set to define two measures of randomness which have interesting properties. If we start with an arbitrary waveform u(t) with Fourier transform U(f), we first calculate the associated Wigner-Moyal distribution function and proceed to define the measure M_1 of that waveform as the joint entropy of the corresponding Wigner-Moyal distribution. M_1 is a positive number that can't be any smaller than log e/2, but it can be arbitrarily large. We define a second measure M_2, closely related to the first, as the determinant of the covariance matrix of the Wigner-Moyal distribution. M_2 is also positive and can't be any smaller than $(4\pi)^{-2}$, but it can be arbitrarily large.

The interesting thing about each of these measures is that the measure of the waveform u(t) is the same as the measure of its Fourier transform U(f). Each is a measure in a very global sense. This is intuitively satisfying because a waveform and its Fourier transform are really the same thing. They are simply, in quantum mechanical terms, a state expressed in different coordinates, and if I'm measuring it you want that measurement to reflect the state and not the particular set of coordinates used to describe it.

You will recall the fact that the Wigner-Moyal distribution might have a negative value at some (p,q). If it has negative values, the entropy is not defined. In this case I can't define the measure M_1. However, even if the Wigner-Moyal distribution assumes negative values, I can always define M_2 if the density has finite second moments.

ALTERNATE MEASURES OF RANDOMNESS

EVERY $u(t)$ WITH FOURIER TRANSFORM $U(f)$ HAS AN ASSOCIATED WMD FUNCTION $F(p,q)$

$$M_1\,[u(t)] = M_1\,[U(f)] = H(F)$$

$$\log\left(\frac{e}{2}\right) \leqslant M_1 \leqslant \infty$$

M_1 IS WELL DEFINED WHEN F IS NON-NEGATIVE REGARDLESS OF MOMENTS

$$M_2\,[u(t)] = M_2\,[U(f)] = \det\,\Sigma$$

$$\left(\frac{1}{4\pi}\right)^2 \leqslant M_2 \leqslant \infty$$

M_2 IS WELL DEFINED WHEN F HAS SECOND MOMENTS EVEN WHEN F IS NEGATIVE

A WAVEFORM $u(t)$ AND ITS SPECTRUM $U(f)$ HAVE IDENTICAL MEASURES OF RANDOMNESS

73732-5

Having defined these measures, I want to suggest how to use them. Let's revisit the problem I introduced to motivate this presentation. Remember we had the unknown far-field distribution U(f), and observations of the aperture illumination $\{a_i\}$. I'm proposing that we estimate U(f) on the following basis. I'm going to postulate that U(f) is simple; i.e., it's not very random. It's not a distribution with a lot of fine structure in it. The way I characterize its simplicity or its well orderedness is by ascribing to it a measure of randomness. Now the particular observations that I have, the $\{a_i\}$, are noisy. In general the noisiness might arise from the fact that the aperture is finite, i.e., truncated. Furthermore, I might not even have all the (Nyquist) samples in the aperture. These various kinds of partial data constitute, in a qualitative sense, noise. There might be additive noise coming into my observations due to the fact that there is receiver noise at each of these sample points. What I'm trying to say is that, in general, my observation is noisier than the signal I'm looking for, and the way I propose to estimate the signal is to operate on my data in such a way as to get out an estimate which is consistent with my data, but has minimum randomness.

To say it another way, all we're doing in terms of assumptions about the structure of the problem is to postulate that the function I'm looking for is simple in the sense of one or another of the measures I'm postulating.

ESTIMATION/DETECTION USING MINIMUM RANDOMNESS CRITERION

- UNKNOWN SPECTRUM U(f) HAS HIGH DEGREE OF ORDER, i.e., LOW MEASURE OF RANDOMNESS

- OBSERVATION CONTAINS NOISE
  - TRUNCATION
  - PARTIAL SAMPLING
  - ADDITIVE
  - PHASE REMOVAL

- NOISY OBSERVATION HAS HIGHER MEASURE OF RANDOMNESS

- ESTIMATE U(f) BY MINIMIZING RANDOMNESS MEASURE AND STILL REMAIN CONSISTENT WITH OBSERVATION

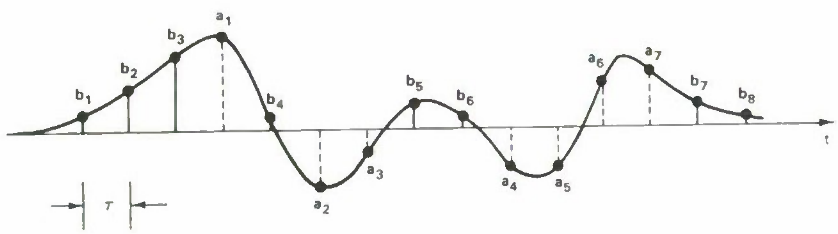
74527-2

Let's see how this point of view can be used to solve a problem. Here is an expanded version of the original problem. There is some unknown waveform u(t), or its spectrum U(f), that I want to estimate. (If I knew u(t) I can calculate U(f).) What I have available is a set of samples of u(t). I call them a_i . I have represented them as not necessarily dense and certainty of finite extent. I want to estimate the missing samples. The missing samples are indicated as $\{b_j\}$. I don't know what they are, but if I have both the known samples and the missing samples together, then I can generate u(t). That u(t) has a certain measure of randomness and I'm going to pick the $\{b_j\}$ such that the resulting u(t) has minimum randomness. That's the point of view. The only thing I'm assuming is that u(t) is band-limited so that I can Nyquist sample it. I pass through the Nyquist sampler. However, I may not have available all the samples, but only a finite number of them. I want to operate on this finite number of samples to guess a set of missing samples $\{b_j\}$ which, when added to the a_i and then sent through an interpolating filter, give the resulting waveform which is simple. By varying the b's I can get either greater or lesser simplicity measured by M_1 or M_2. I'm looking for the set of b's which gives u(t) with minimum M_1 or M_2.

INTERPOLATION/EXTRAPOLATION

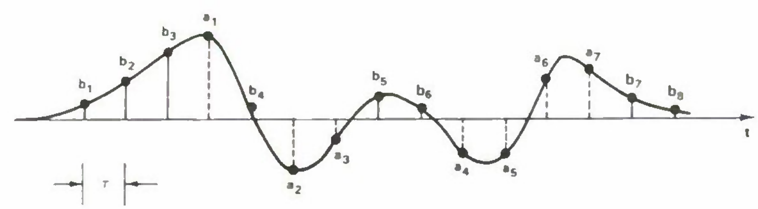
GOAL: RECONSTRUCT WAVEFORM $u(t)$ [OR SPECTRUM $U(f)$] WHEN OBSERVER HAS AVAILABLE ONLY FINITE NUMBER OF SAMPLES OF $u(t)$

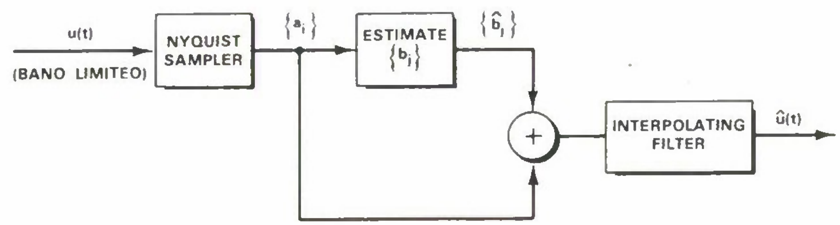
MODEL: AVAILABLE DATA IS $\{a_i\}$ WHERE $a_i = u(t_i)$
WANT TO ESTIMATE $\{b_j\}$ WHERE $b_j = u(t_j)$
τ = NYQUIST SAMPLING INTERVAL



74527-3

INTERPOLATION/EXTRAPOLATION — Continued





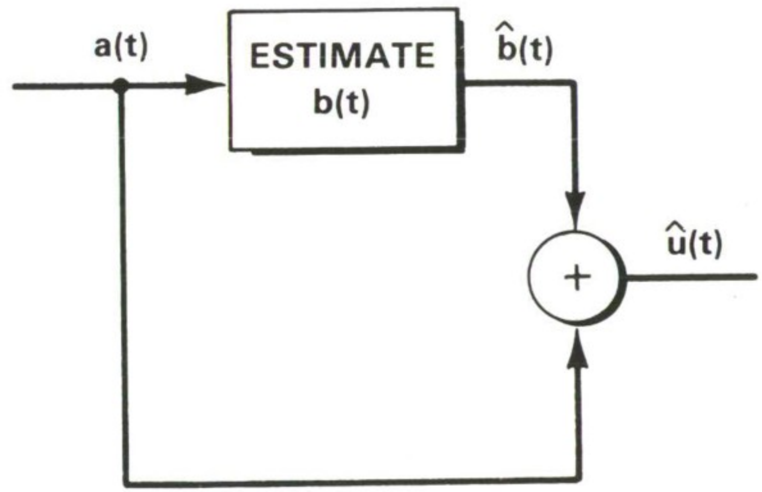
ESTIMATE $\{b_j\}$ SO THAT $\hat{u}(t)$ HAS MINIMUM RANDOMNESS MEASURE

74527-4

A second class of problems that might be addressed by this point of view is suggested in the next cartoon. This is another problem we are frequently asked to solve in which there is some unknown signal that may be imbedded in noise and we receive the noisy composite waveform a(t). Maybe there's a signal there, maybe there isn't. We don't know anything about the signal. We really don't know very much about the noise either and what we want to do is "optimally" estimate what u(t) is, consistent with our observation of a(t). This is related to non-parametric estimation methods in which we don't want to make assumptions about the statistics or the noise or, if we do make assumptions about the statistics, we don't want to specify parameters like means, variances, covariances, etc. So, symbolically at least, we have this observation a(t) and we want to estimate the unknown b(t), add it to the u(t) such that what comes out is simple in terms of the suggested measure.

NON-PARAMETRIC EXTRACTION OF UNKNOWN SIGNAL FROM NOISE

GOAL: EXTRACT SIGNAL $u(t)$ FROM RECEIVED SIGNAL $a(t) = u(t) + n(t)$ WITH MINIMAL ASSUMPTIONS ABOUT $u(t)$ AND $n(t)$



ESTIMATE $b(t)$ SO THAT $\hat{u}(t) = a(t) + \hat{b}(t)$ HAS MINIMUM RANDOMNESS MEASURE

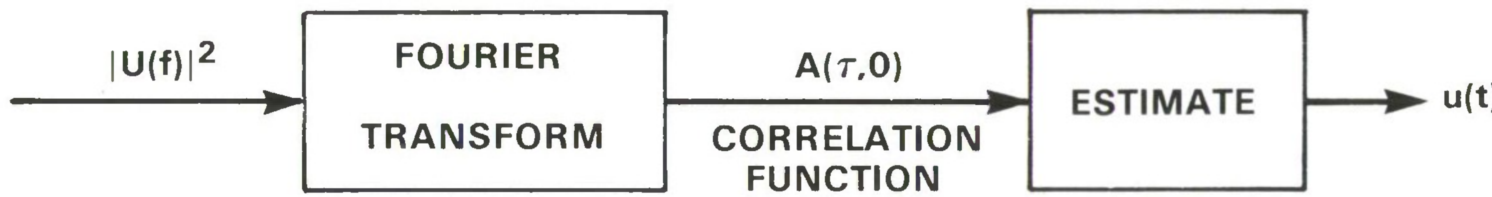
73732-8

A third class of problem we address is the estimation of a complex waveform when only its square magnitude is known. This is an idealization of a situation in which a detector array in the focal plane of an imaging system senses only energy (not phase), and we desire to estimate the special distribution of sources which generated the focal plane distribution.

The problem is solved by selecting that waveform with the specified square magnitude which has the minimum measure of randomness.

PHASE RETRIEVAL

GOAL: ESTIMATE SIGNAL $u(t)$ [OR SPECTRUM $U(f)$] WHEN OBSERVER HAS AVAILABLE ONLY MAGNITUDE INFORMATION $|u(t)|^2$ (OR $|U(f)|^2$)



ESTIMATE $u(t)$ AS THAT WAVEFORM WITH CORRELATION FUNCTION $A(\tau,0)$ HAVING MINIMUM RANDOMNESS MEASURE

74527-1

I have indicated three classes of problems to which the suggested point of view might be applied. I haven't yet worked any examples. I'm taking this opportunity to describe the point of view.

I want to close by going back on my checklist of desirable attributes of measures and offer some commentary on them.

Are my suggested measures intuitively pleasing? I think they are intuitively pleasing because the measures are global. They do not depend on fine structure. They do not have buried in them lots of assumptions that very often are unjustified.

Both measures are robust in the sense that they are very well-behaved functions.

Do the measures give rise to tractable solutions? Probably not, although there aren't very many optimization methods that are. Whether this measure is more or less tractable than others remains to be seen.

Is the optimum unique? I don't really know, but I offer a hopeful conjecture. The hope is supported by the fact that the entropy function that is suggested as one of the measures has very interesting convexity properties. It's convex in all of its variables and there is at least a chance that it will converge to a single optimum instead of several in many practical problems.

I'll repeat the fact that I haven't worked any examples and that I don't know if the suggested approach is really useful. One of the primary purposes I have in presenting this to you is to expose the point of view to the Laboratory's technical community.

FINAL REMARKS

INTUITIVELY PLEASING?	YES
ARBITRARY ASSUMPTIONS?	MINIMAL
ROBUST?	YES
TRACTABLE?	PROBABLY NOT
UNIQUE OPTIMUM?	NOT KNOWN. HOWEVER, ENTROPY FUNCTION HAS STRONG CONVEXITY PROPERTIES

NO EXAMPLES WORKED

UTILITY TBD

73732-9

REFERENCES

1. E. Wigner, "Quantum Correction for Thermodynamic Equilibrium," Physical Review **40**, 749-760 (1932).
2. J. E. Moyal, "Quantum Mechanics as a Statistical Theory," Cambridge Phil. Society **45**, 99-124 (1949).
3. R. Leipnik, "The Extended Entropy Uncertainty Principle," Information and Control **3**, 18-25 (1960).
4. P.M. Woodward, *Probability and Information Theory, with Applications to Radar* (McGraw-Hill, New York, 1955).

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*This report is an edited version of a Laboratory seminar delivered by the author on 14 February 1986. The format displays the text associated with the vugraphs used during the seminar.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

entropy
randomness of signals
uncertainty principle
quantum mechanics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The solution to an optimization or estimation problem depends on some criterion of optimality. Typically, all prospective solutions have an associated measure of goodness, and the optimum solution is that with maximum (or minimum) measure. The choice of measure incorporates whatever is known about the physics of the problem as well as other assumptions.

In this presentation, we suggest two closely related measures of the randomness of signals. Each of these measures is related to the uncertainty principle of quantum mechanics. We further suggest that minimizing either of these measures is an appropriate criterion for the solution of a variety of problems, including spectral estimation, signal extrapolation and/or interpolation, non-parametric signal detection, and phase retrieval. The criterion may be appropriate when the unknown signal is postulated to be "simple" rather than "complex" and we want to minimize the number of arbitrary assumptions made about the signal and noise characteristics.